CLASSIFICATION OF SOLUTIONS TO TODA SYSTEMS OF TYPES C AND B WITH SINGULAR SOURCES

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Abstract. In this paper, the classification in [LWY12] of solutions to Toda systems of type A with singular sources is generalized to Toda systems of types C and B. Like in the A case, the solution space is shown to be parametrized by the abelian subgroup and a subgroup of the unipotent subgroup in the Iwasawa decomposition of the corresponding complex simple Lie group. The method is by studying the Toda systems of types C and B as reductions of Toda systems of type A with symmetries. The theories of Toda systems as integrable systems as developed in [Lez80, LS92, Nie12, Nie14], in particular the W-symmetries and the iterated integral solutions, play essential roles in this work, together with certain characterizing properties of minors of symplectic and orthogonal matrices.

1. Introduction

The Toda systems on the plane with singular sources that we will consider in this paper are the elliptic versions of the conformal Toda field theories. Toda filed theories are fundamental examples of integrable systems and have many applications in mathematics and physics. Toda field theories have been widely studied in the literature using various viewpoints and techniques, and we refer the reader to a very partial list of [Lez80, LS92, BFO+90, Nie12, Nie14]. The elliptic Toda systems are closely related to the non-relativistic and relativistic non-abelian Chern-Simons gauge field theory [Yan01]. In particular, the results of this paper will be useful in constructing non-topological solutions to the relativistic Chern-Simons gauge field theory, following the works of [CI00, ALW13] where the cases of A_1, A_2 and C_2 are treated.

Let us introduce the systems that we will consider following the fundamental work [LWY12]. To each complex simple Lie algebra there is associated a Toda system. A complex simple Lie algebra $\mathfrak{g}$ of rank n is classified by its Cartan matrix (a_{ij}), a rank-n matrix with integer entries such that $a_{ii} = 2$ and $a_{ij} \leq 0$ when $i \neq j$. The classification result of Killing and Cartan states that the complex simple Lie algebras come in four classical series called A_n, B_n, C_n and D_n with the respective simple Lie algebras being $\mathfrak{sl}_{n+1}$, $\mathfrak{so}_{2n+1}$, $\mathfrak{sp}_{2n}$, $\mathfrak{so}_{2n}$, together with five exceptional Lie algebras $G_2, F_4, E_{6,7,8}$. We refer the reader to, for example, [FH91, Kna02] for basic Lie theory.

To each simple Lie algebra $\mathfrak{g}$ of rank n with Cartan matrix $(a_{ij})_{i,j=1}^n$, we consider the following associated Toda system on the plane with singular sources at the origin

and with finite integrals

$$
(1.1) \qquad
\begin{cases}
\Delta u_i + 4 \displaystyle\sum_{j=1}^{n} a_{ij} e^{u_j} = 4\pi \gamma_i \delta_0, & \gamma_i > -1, \\[2mm]
\displaystyle\int_{\mathbb{R}^2} e^{u_i}\, dx < \infty, & 1 \le i \le n.
\end{cases}
$$

Here the u_i are real-valued functions on the plane with coordinates $x = (x_1, x_2)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, and the δ_0 is the Dirac delta function at the origin. The work [LWY12] is the first to systematically study Toda systems with singular sources at the origin and with the finite integral conditions. Such a setup leads to the powerful classification result in [LWY12] and is useful in studying the non-topological solutions of the relativistic Chern-Simons gauge theory in [ALW13].

In solving (1.1), we will heavily use the complex coordinates $z = x_1 + i\, x_2$ and $\bar{z} = x_1 - i\, x_2$. For simplicity, we write $\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})$, and similarly $\partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})$. The Laplace operator is then

$$
(1.2) \qquad\qquad\qquad \Delta = 4\partial_z \partial_{\bar{z}}.
$$

The coefficient 4 here is responsible for the slightly unconventional coefficient 4 on the left of (1.1). For more on this, see Remark 1.12 at the end of this introduction.

The local solutions of the systems (1.1), when viewed using (1.2), have been well studied in terms of integrable systems especially by the works of Leznov and Saveliev [Lez80, LS79, LS92]. Given n arbitrary functions $\phi_i(z)$ of z and n arbitrary functions $\psi_i(\bar{z})$ of $\bar{z}$, local complexed-valued solutions $u_1, \ldots, u_n$ can be constructed using some representation theory of the simple Lie group G corresponding to the Cartan matrix (a_{ij}). The requirements of global real solutions with finite integrals in (1.1) will put on rigid restrictions on these arbitrary functions, and the difficulty of the classification lies in proving that all the solutions to (1.1) are of certain forms.

In the literature, general classification results of the global solutions to the Toda systems (1.1) with finite integrals only exists for Lie algebras of type A, although [ALW13] and [ALW15] have treated the C_2 and G_2 cases. Such classification results started without the singular sources, that is, when $\gamma_i = 0$. When the Lie algebra is $A_1 = \mathfrak{sl}_2$, the system (1.1) becomes the ubiquitous Liouville equation, and Chen and Li [CL91] classified their finite-integral global solutions by the moving plane method following [GNN81]. Jost and Wang [JW02] extended the classification to the general A_n cases without singular sources using some theorem from algebraic geometry about totally unramified curves. In the thorough work [LWY12], Lin, Wei and Ye established the complete classification for general A_n systems with singular sources together with the non-degeneracy properties for such solutions. One ingenious step in their proofs is to control the solutions by an ODE using the W-symmetries of the Toda systems, which are shown to take simple forms in the current setup. This author has studied the W-symmetries in [Nie14] under the name of characteristic integrals.

In this paper, we obtain the complete classification of the solutions to the system (1.1) for Lie algebras of types C and B. Our convention is that the Cartan matrices

for A_n, C_n and B_n are respectively

(1.3) $A_n :$ $\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$,

(1.4) $C_n :$ $\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}$,

(1.5) $B_n :$ $\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -2 \\ & & & -1 & 2 \end{pmatrix}$.

First we cast the classification result of [LWY12] in the form that we would like to generalize, and we refer the reader to [Kna02, Hel78] for the Iwasawa decomposition.

Theorem 1.6 ([LWY12])**.** *For the Lie algebra A_n, the corresponding simple complex Lie group is $G = SL(n+1, \mathbb{C})$. Let $G = KAN$ be its Iwasawa decomposition, where $K = SU(n+1)$ is compact, $A = \mathbb{R}_+^n$ is abelian, and N is the unipotent subgroup of unipotent lower-triangular matrices. The space of solutions to (1.1) of type A_n is parametrized by AN_Γ, where N_Γ is a subgroup of N determined by the set of γ_i. In particular, if all the γ_i are integers, then $N_\Gamma = N$ and the dimension of the solution space is the dimension of the space AN, which is $n^2 + 2n$. Furthermore there are concrete formulas for the solution associated to an element in the relevant space AN_Γ.*

Our main theorem is that similar results continue to hold in the C_n and B_n cases, and it follows from Section 4 and Propositions 5.5, 5.21, and 6.6.

Theorem 1.7.   (i)   *For the Lie algebra C_n, the corresponding simple complex Lie group is $G = Sp(2n, \mathbb{C})$, the group of symplectic matrices preserving J_{2n} in (3.1). Let $G = KAN$ be its Iwasawa decomposition, where $K = Sp(2n)$ is compact, $A = \mathbb{R}_+^n$ is abelian, and N is the unipotent subgroup of unipotent lower-triangular matrices in $Sp(2n, \mathbb{C})$. The space of solutions to (1.1) of type C_n is parametrized by AN_Γ, where N_Γ is a subgroup of N determined by the set of γ_i. In particular, if all the γ_i are integers, then $N_\Gamma = N$ and the dimension of the solution space is the dimension of the space AN, which is $2n^2 + n$. Furthermore there are concrete formulas for the solution associated to an element in the relevant space AN_Γ.*

(ii)   *For the Lie algebra B_n, the corresponding simple complex Lie group is $G = SO(2n+1, \mathbb{C})$, the group of special orthogonal matrices preserving J_{2n+1} in (3.1). Let $G = KAN$ be its Iwasawa decomposition, where $K = SO(2n+1)$ is compact, $A = \mathbb{R}_+^n$ is abelian, and N is the unipotent subgroup of unipotent lower-triangular matrices in $SO(2n+1, \mathbb{C})$. The space of solutions to (1.1) of type B_n is*

parametrized by AN_Γ, where N_Γ is a subgroup of N determined by the set of γ_i. In particular, if all the γ_i are integers, then $N_\Gamma = N$ and the dimension of the solution space is the dimension of the space AN, which is $2n^2 + n$. Furthermore there are concrete formulas for the solution associated to an element in the relevant space AN_Γ.

The system (1.1) has another version which is easier to use for many purposes. Let $u_i = \sum_{j=1}^n a_{ij} U_j$. Then the U_i satisfy

$$(1.8) \qquad \begin{cases} \Delta U_i + 4 \exp \Big(\sum_{j=1}^n a_{ij} U_j \Big) = 4\pi \alpha_i \delta_0 \\ \int_{\mathbb{R}^2} e^{\sum_j a_{ij} U_j} \, dx < \infty, \qquad 1 \le i \le n. \end{cases}$$

Here

$$(1.9) \qquad \alpha_i = \sum_{j=1}^n a^{ij} \gamma_j,$$

where (a^{ij}) is the inverse matrix of (a_{ij}). By (1.2), the first equation in (1.8) is the same as

$$(1.10) \qquad U_{i,z\bar{z}} + \exp \Big(\sum_{j=1}^n a_{ij} U_j \Big) = \pi \alpha_i \delta_0,$$

and it is this form that is usually called Toda field theory.

We emphasize that the main tools in this paper come from the theory of the Toda systems as integrable systems as developed by Leznov and Saveliev [Lez80, LS79, LS92], with some further development and clarification by this author [Nie12, Nie14]. Using the structure of W-symmetries in [Nie14], we can derive some of the results in [LWY12] concerning Toda systems of type A more conceptually and quickly. Furthermore the results in [LWY12] are best presented using the iterated integral viewpoint explicitly spelled out in [Lez80]. The results in Theorem 1.7 are obtained using the structure theory of the solutions to Toda systems of types C and B in [Nie12]. It is a well-known fact that the Toda systems of type C_n and B_n are reductions of those of types A_{2n-1} and A_{2n}. Therefore the results in [LWY12] for type A lead to results for type C_n and B_n. However without the correct viewpoint, this reduction procedure becomes tedious and un-illuminating. In the literature, [ALW13, ALW15] worked out such reductions for the cases of C_2 and G_2 with long formulas. In this paper, we show that with the correct viewpoint, the results are still expressed using the nice structure of Lie groups.

This paper is organized as follows. In Section 2, we clarify some key results of [LWY12] using the structure theory in [Nie14], and cast the results in [LWY12] in the framework using iterated integrals [Lez80]. In Section 3, we establish some characterizing algebraic properties of minors of symplectic and orthogonal matrices. Here and throughout the paper, we are able to treat the symplectic and the orthogonal cases on the same footing by the bilinear form in (3.1). In Section 4, we spell out the C_n and B_n systems as reductions of A_k systems with symmetries. Then finally in Section 5, we show that the symmetry reduction requirement forces the group elements in the solutions to the A_k system to be more special, that is, they are symplectic or orthogonal respectively. Finally in Section 6, we study the subgroup N_Γ of N determined by the γ_i for the solutiosn to be well-defined on

$\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. In the appendices, we show the examples of C_3 and B_2 Toda systems to demonstrate our results.

Remark 1.11. Toda systems (1.1) for Lie algebras of types D, E and F can't be studied along the line of this paper since they are not reductions of the type A. Their study will be pursued in a future work.

Remark 1.12. The coefficients 4 on the left hand sides of (1.1) and (1.8) come from (1.2). With this coefficient 4, we get the most convenient form of equation (1.10) to which we will apply many tools from Toda field theories. This coefficient can be easily dealt with as follows.

The solutions u_i to (1.1) and the solutions $\hat{u}_i$ to the more conventional equation in [LWY12], $\Delta\hat{u}_i + \sum_{j=1}^n a_{ij}e^{\hat{u}_j} = 4\pi\gamma_i\delta_0$, are related by

$$\hat{u}_i = u_i + \ln 4, \quad 1 \le i \le n.$$

Similarly the solutions U_i to (1.8) and the solutions $\hat{U}_i$ to the more conventional equation in [LWY12], $\Delta\hat{U}_i + \exp\left(\sum_{j=1}^n a_{ij}\hat{U}_j\right) = 4\pi\alpha_i\delta_0$, are related by

$$\hat{U}_i = U_i + (\ln 4)\sum_{j=1}^n a^{ij}, \quad 1 \le i \le n.$$

Our current choice of coefficients makes many formulas easier. For example, the coefficient $2^{-n(n+1)}$ in Eq. (1.11) of [LWY12] would be gone under our convention.

2. THE A CASE REVISITED USING TODA FIELD THEORIES

In this section, we revisit the fundamental classification result in [LWY12] for solutions to Toda systems (1.1) with singular sources for Lie algebras of type A, that is, with Cartan matrix (1.3). We apply the theory of Toda systems as integrable systems to reformulate some of their proofs and results.

For the reader's convenience, first we recall the classification result in [LWY12] for type A. For Toda systems (1.8) of types A, C, B, and G_2, one simplification is that U_1 decides all the other U_i by the shapes of their Cartan matrices.

Theorem 2.1 ([LWY12]). *For the system (1.8) of type A_n, the U_1 is expressed by*

(2.2)
$$e^{-U_1} = |z|^{-2\alpha_1}\left(\lambda_0 + \sum_{i=1}^n \lambda_i|P_i(z)|^2\right), \quad where$$

$$P_i(z) = z^{\mu_1 + \cdots + \mu_i} + \sum_{j=0}^{i-1} c_{ij}z^{\mu_1 + \cdots + \mu_j},$$

$\mu_i = 1 + \gamma_i > 0$, c_{ij} are complex numbers, and the $\lambda_i > 0$ satisfy

(2.3)
$$\lambda_0 \cdots \lambda_n = \prod_{1 \le i \le j \le n}\left(\sum_{k=i}^j \mu_k\right)^{-2}.$$

Furthermore for $j < i$,

(2.4) $c_{ij} = 0$ *if $\mu_{j+1} + \cdots + \mu_i \notin \mathbb{N}$.*

The major tool in [LWY12] is the following characterization of the components of e^{-U_1}. This is proved in Steps 1 and 2 of the proof of the main theorem 5.1 in [LWY12].

Proposition 2.5 ([LWY12])**.** *For the system (1.8) of type A_n, the U_1 is expressed by*

$$ (2.6) \qquad e^{-U_1} = \sum_{i,j=0}^{n} m_{ij} \bar{f}_i f_j, $$

where $M = (m_{ij})_{i,j=0}^{n}$ is a Hermitian matrix, and $f_i(z) = z^{\beta_i}$ with

(2.7) $\beta_0 = -\alpha_1,$ $\beta_i = \alpha_i - \alpha_{i+1} + i$ *for $1 \le i \le n-1$,* $\beta_n = \alpha_n + n$.

New proof of Prop 2.5 using [Nie14]. The strategy for the proof is as follows. For the reader's convenience, we first repeat some estimates from [LWY12] using [BM91]. Then we present the formulas in [Nie14] for computing the W-symmetries, also called characteristic integrals W_j. Then we use the above estimates to quickly show that the W_j have simple forms, and this further implies that e^{-U_1} has simple components.

Following [LWY12, Eq. (5.10)], introduce

(2.8) $V_i = U_i - 2\alpha_i \log|z|,$ $1 \le i \le n.$

Then system (1.8) becomes

$$ \begin{cases} \Delta V_i = -4|z|^{2\gamma_i} \exp\Big(\sum_{j=1}^{n} a_{ij} V_j \Big) \\ \displaystyle\int_{\mathbb{R}^2} |z|^{2\gamma_i} \exp\Big(\sum_{j=1}^{n} a_{ij} V_j \Big) dx < \infty. \end{cases} $$

As $\gamma_i > -1$, applying Brezis-Merle's argument in [BM91], we have $V_i \in C^{0,\alpha}$ on $\mathbb{C}$ for some $\alpha \in (0,1)$ and they are upper bounded over $\mathbb{C}$. Furthermore

$$ (2.9) \qquad \begin{aligned} &\partial_z^k V_i(z) = O(1 + |z|^{2+2\gamma_i-k}) \quad \text{near } 0, \\ &\partial_z^k V_i(z) = O(|z|^{-k}) \quad \text{near } \infty, \forall\, k \ge 1. \end{aligned} $$

Now (2.8) clearly implies that

$$ (2.10) \qquad \begin{aligned} &\partial_z^k U_i(z) = O(|z|^{-k}) \quad \text{near } 0, \forall\, k \ge 1, \\ &\partial_z^k U_i(z) = O(|z|^{-k}) \quad \text{near } \infty, \forall\, k \ge 1. \end{aligned} $$

For the Toda system (1.8) of type A, the W-symmetries are computed in [Nie14, Example 3.1] by the expansion

$$ \mathcal{L} = (\partial_z - U_{n,z})(\partial_z + U_{n,z} - U_{n-1,z}) \cdots (\partial_z + U_{2,z} - U_{1,z})(\partial_z + U_{1,z}) $$

(2.11)
$$ = \partial_z^{n+1} + \sum_{j=1}^{n} W_j \partial_z^{n-j}. $$

(Here we have used the symmetry of the A_n system with respect to U_i and U_{n+1-i} in the original formula (3.1) in [Nie14] to conform to [LWY12]. For $1 \le j \le n$, our

$W_j = Z_{n+1-j}$ in [LWY12, Lemma 5.2].) The W_j are differential polynomials in the U_i, that is, they are polynomials in the U_i and their derivatives with repsect to z. The W_j are called characteristic integrals and denoted by I_j in [Nie14], since for solutions U_i to the Toda system (1.8), they satisfy the property

$$(2.12) \qquad W_{j,\bar{z}} = 0, \quad 1 \le j \le n.$$

The W_j are also called W-symmetries in [LWY12]. For a differential mononomial in the U_i, we call by its *degree* the sum of the orders of differentiation multiplied by the algebraic degrees of the corresponding factors. For example $U_{1,zz} - U_{1,z}^2$ has a homogeneous degree 2. Clearly from the formula (2.11), we see that W_j has homogeneous degree $j + 1$ for $1 \le j \le n$.

Therefore (2.10) and (2.12) imply that $z^{j+1}W_j$ is holomorphic and bounded on $\mathbb{C}^*$, and hence is a constant by Liouville's theorem. That is, for $1 \le j \le n$,

$$(2.13) \qquad W_j = \frac{w_j}{z^{j+1}} \quad \text{on } \mathbb{C}^*$$

for some number w_j.

Since $\gamma_i > -1$, Eqs. (2.9) and (2.10) imply that near 0, the W_j can be computed using the

$$2\alpha_i \log |z| = \alpha_i (\log z + \log \bar{z})$$

summand of U_i in (2.8). That is, we have

$$(2.14) \qquad \begin{aligned} &\left(\partial_z - \frac{\alpha_n}{z}\right)\left(\partial_z + \frac{\alpha_n - \alpha_{n-1}}{z}\right) \cdots \left(\partial_z + \frac{\alpha_2 - \alpha_1}{z}\right)\left(\partial_z + \frac{\alpha_1}{z}\right) \\ &= \partial_z^{n+1} + \sum_{j=1}^{n} W_j \partial_z^{n-j}. \end{aligned}$$

This totally determines the $W_j = \frac{w_j}{z^{j+1}}$, and in particular the w_j as real numbers. Now from (2.11), it is clear that $\mathcal{L}e^{-U_1} = 0$. Call $f = e^{-U_1}$, and we have

$$f^{(n+1)} + \sum_{j=1}^{n} \frac{w_j}{z^{j+1}} f^{(n-j)} = 0.$$

The fundamental solutions of this Cauchy-Euler equation are of the form z^{β_i} for $0 \le i \le n$. We use (2.14) to quickly see that the β_i are those specified in (2.7).

Since we require that our solutions are real, we see that e^{-U_1} must have the form in (2.6) using a Hermitian matrix. $\qquad \square$

Remark 2.15. [LWY12] obtained the characteristic exponents in (2.7) by constructing the solutions to the Toda system (1.8) and hence their method is indirect. Here the structure theory of the W-symmetries developed by this author makes this proof more transparent. Actually, in Remark 3.4 of [LWY12], the authors mentioned that the explicit formulas of these invariant are hard to find in the literature, and they constructed these invariants by a different method in [LWY12].

In the remainder of this section, we cast the results in [LWY12] in the framework of iterated integral solutions of [Lez80]. There are no new results or proofs here, except that the viewpoint is to this author more conceptual. Using $\mu_i = \gamma_i + 1$ for $1 \le i \le n$, it is easy to see that (2.7) is equivalent to

$$(2.16) \qquad \beta_0 = -\alpha_1, \quad \beta_i - \beta_0 = \mu_1 + \cdots + \mu_i, \ \forall 1 \le i \le n.$$

In the literature of Toda systems as integrable systems, especially in [Lez80], there is a general construction of solutions to the Toda systems of type A given n arbitrary functions $\phi_1(z), \ldots, \phi_n(z)$ of z and n arbitrary functions of $\bar{z}$. First we formally and locally define the intreated integrals

$$(2.17) \quad \sigma_0(z) := 1, \; \sigma_1(z) := \int_0^z \phi_1(z_1) \, dz_1,$$

$$\sigma_i(z) := \int_0^z \phi_1(z_1) \, dz_1 \cdots \int_0^{z_{i-2}} \phi_{i-1}(z_{i-1}) \, dz_{i-1} \int_0^{z_{i-1}} \phi_i(z_i) \, dz_i, \; \text{for a general } i.$$

Also define

$$(2.18) \qquad\qquad \xi(z) := \prod_{j=1}^n \phi_j(z)^{a^{1j}} = \prod_{j=1}^n \phi_j(z)^{\frac{n+1-j}{n+1}},$$

where the $a^{1j} = \frac{(n+1-j)}{n+1}$ are from the first row of the inverse Cartan matrix to (1.3). Define

$$\nu_i(z) := \frac{\sigma_i(z)}{\xi(z)}, \quad 0 \le i \le n.$$

The important property [Lez80, LS92, Nie14] is that the Wronskian

$$(2.19) \qquad\qquad W(\nu_0, \nu_1, \ldots, \nu_n) = 1$$

for n arbitrary functions $\phi_1, \cdots, \phi_n$.

Since we are only interested in real solutions, the functions of $\bar{z}$ are hence taken to be just the conjugates. Then [Lez80] asserts that

$$(2.20) \qquad e^{-U_1} = |\nu_0|^2 + |\nu_1|^2 + \cdots + |\nu_n|^2 = \frac{1 + |\sigma_1(z)|^2 + \cdots + |\sigma_n(z)|^2}{\left|\xi(z)\right|^2}$$

defines a solution to the Toda systems of type A_n.

Very neatly put, Proposition 2.5 through the W-symmetries just determines that the integrand functions in the iterated integral scheme of Leznov and Saveliev [Lez80, LS79, LS92] are just the following functions

$$(2.21) \qquad\qquad \phi_i(z) = z^{\gamma_i}, \quad \forall 1 \le i \le n.$$

Then (2.17) and (2.18) become (at least after the branch cut on $\mathbb{C} \backslash \{x \,|\, x_1 \le 0\}$)

$$(2.22) \qquad \sigma_i(z) = \frac{z^{\mu_1 + \cdots + \mu_i}}{\mu_i(\mu_i + \mu_{i-1}) \cdots (\mu_i + \cdots + \mu_1)}, \quad \forall 0 \le i \le n,$$

and

$$(2.23) \qquad\qquad \xi(z) = \prod_{j=1}^n z^{a^{1j}\gamma_j} = z^{\alpha_1}.$$

Using (2.16), we see that

$$(2.24) \qquad \nu_i(z) = \frac{\sigma_i(z)}{\xi(z)} = \frac{1}{\mu_i \cdots (\mu_i + \cdots + \mu_1)} z^{\beta_i} = \chi_i f_i, \quad 0 \le i \le n.$$

That is, ν_i is the same as the f_i in Proposition 2.5 up to a scale $\chi_i = \prod_{j=1}^i \left(\sum_{k=j}^i \mu_k\right)^{-1}$.

Therefore (2.20) becomes

$$(2.25) \qquad e^{-U_1} = |z|^{-2\alpha_1} \Big(\chi_0^2 + \sum_{i=1}^{n} \chi_i^2 |z^{\mu_1 + \cdots + \mu_i}|^2 \Big).$$

We note that corresponding to (2.3), we have

$$\chi_0^2 \cdots \chi_n^2 = \prod_{1 \le i \le j \le n} \Big(\sum_{k=i}^{j} \mu_k \Big)^{-2}.$$

The solution (2.25) corresponds to a radial solution of the Toda system (1.8), and there are other solutions. The extra freedom comes from the Hermitian matrix M in terms of the f_i from (2.6). From our point of view, the version using the ν_i in (2.24) is more natural and has better properties later on. Therefore we define

$$(2.26) \qquad e^{-U_1} = \sum_{i,j=0}^{n} h_{ij} \bar{\nu}_i \nu_j,$$

where $H = (h_{ij})_{i,j=0}^{n} = XMX$, and $X := \mathrm{diag}(\chi_0, \chi_1, \cdots, \chi_n)^{-1}$ is the diagonal matrix.

Then H is Hermitian and must have determinant 1, i.e. $H \in SL(n+1, \mathbb{C})$, for (2.26) to define a solution of the Toda system (1.8) of type A_n. See [Lez80, Nie12], and this corresponds to condition (2.3) from [LWY12].

Furthermore as shown in [LWY12], M and hence H must be positive definite. The Cholesky decomposition [GVL96] then states that

$$(2.27) \qquad H = B^\dagger B,$$

where B is a lower-triangular matrix with diagonal entries $b_{ii} > 0$ and $B^\dagger$ denotes the conjugate transpose $\bar{B}^t$. Clearly $B = \Lambda C$ where Λ is the diagonal of B and C is lower-triangular with 1's on its diagonal.

This then gives the form (2.2) of the solutions in Theorem 2.1. For the solution to be well-defined on $\mathbb{C}^*$, it is easy to see the condition (2.4) on the c_{ij}.

We summarize the results for the A_n case in the following form, which provides more details to Theorem 1.6.

Proposition 2.28. *All the solutions to the Toda system* (1.8) *of type A_n are given, in terms of the first unknown U_1, in the following way. Using the family of functions $\phi_i(z)$ as in* (2.21), *define the iterated integrals $\sigma_i(z)$ as in* (2.22) *and the $\nu_i(z) = \frac{\sigma_i(z)}{\xi(z)}$ as in* (2.24) *with $\xi(z) = z^{\alpha_1}$ as in* (2.23). *Consider the vector of functions*

$$(2.29) \qquad \boldsymbol{\nu} = (\nu_0, \nu_1, \cdots, \nu_n)^t.$$

Let $G = SL_{n+1}(\mathbb{C})$ and let $G = KAN$ be its Iwasawa decomposition with $K = SU(n+1)$, A the abelian group of diagonal matrices with positive diagonal entries and determinant 1, and N the unipotent group of unipotent lower-triangular matrices. Corresponding to (2.4)*, let N_Γ be the subgroup of N corresponding to the following system of roots $\Delta_\Gamma \subset \Delta^+$*

$$(2.30) \qquad \Delta_\Gamma := \Big\{ \sum_{k=j+1}^{i} \tau_k \ \Big| \ \sum_{k=j+1}^{i} \gamma_k \in \mathbb{Z}, \ j < i \Big\}.$$

Here the τ_k are the system of simple roots for A_n, and this system Δ_Γ is closed under addition. (See Definition 6.3 for more.)

Then the solution space of (1.8) *is parametrized by* AN_Γ. *In particular for* $\Lambda \in A$ *and* $C \in N_\Gamma$,

$$U_1 = -\log |\Lambda C \boldsymbol{\nu}|^2$$

defines a solution.

3. Minors of symplectic and orthogonal matrices

In preparation for our classification of solutions to the C_n and B_n Toda systems, we need some algebraic properties of symplectic and odd-dimensional orthogonal matrices. More precisely, we will show that such matrices in the special linear group are characterized by some equalities among their minors.

First we define the following rank-k matrix

(3.1)
$$J_k = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & \iddots & & & \\ (-1)^{k-1} & & & & \end{pmatrix},$$

with ± 1 along the secondary diagonal. When $k = 2n$ is even, J_{2n} is skew-symmetric and defines a non-degenerate skew-symmetric bilinear form on $\mathbb{C}^{2n}$. When $k = 2n+1$ is odd, J_{2n+1} is symmetric and defines a non-degenerate symmetric bilinear form on $\mathbb{C}^{2n+1}$. Note that

(3.2)
$$J^{-1} = (-1)^{k-1}J.$$

We define the complex symplectic group and complex special orthogonal group in dimensions $2n$ and $2n+1$ with respect to the J_{2n} or J_{2n+1}, that is,

(3.3)
$$Sp(2n, \mathbb{C}) := \{A \in GL(2n, \mathbb{C}) \,|\, A^t J_{2n} A = J_{2n}\}$$

(3.4)
$$SO(2n+1, \mathbb{C}) := \{A \in SL(2n+1, \mathbb{C}) \,|\, A^t J_{2n+1} A = J_{2n+1}\}.$$

A matrix A in $Sp(2n, \mathbb{C})$ is called a symplectic matrix, and it is well-known that $\det A = 1$. A matrix A in $SO(2n+1, \mathbb{C})$ is required to have $\det A = 1$ (in general the determinant is ± 1), and is called a special odd-dimensional orthogonal matrix, or just an orthogonal matrix for short.

Remark 3.5. There are other more conventional choices of the matrices defining the bilinear forms, and our versions can be related to them easily. For example in the symplectic case, to transform from J_{2n} in (3.1) to the more conventional

$$\Omega_{2n} = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix},$$

we define a diagonal matrix Q with

(3.6)
$$Q_{ii} = \begin{cases} (-1)^{i-1} & 1 \le i \le n \\ 1 & n+1 \le i \le 2n \end{cases},$$

and then

$$QJ_{2n}Q = \Omega_{2n}.$$

Therefore if A in (3.3) is symplectic with respect to J_{2n}, then QAQ is symplectic with respect to Ω_{2n}.

Similarly in our orthogonal case, the Q in (3.6) with the $2n$ replaced by $2n+1$ transforms our J_{2n+1} in (3.1) to the more conventional secondary diagonal matrix Θ with $\Theta_{i,2n+2-i} = 1$ for $1 \leq i \leq 2n+1$.

This author discovered the bilinear form in (3.1) in the work [Nie12], and they are the most relevant for the study of Toda systems as they are naturally related to the differential properties of iterated integrals (see Proposition 5.10 below). Furthermore the related algebraic properties of minors are also nicer since no signs come up at all. Proposition 3.12 below is an improvement of [Nie12, Prop. 3.4], which only treated the case of principal minors. The corresponding result using the more conventional bilinear forms for general minor would involve many signs that are hard to pin down (see [Nie12, Remark 3.5]).

Let us first introduce some notation. Let $\{e_i\}_{i=1}^{k}$ be the standard basis of $\mathbb{C}^k$. Let $A \in SL(k,\mathbb{C})$ be a matrix with determinant 1. We are concerned with the cases where $k = 2n$ and A is symplectic as in (3.3), or $k = 2n+1$ and A is special orthogonal as in (3.4). The matrix $A = (a_{ij})$ defines a linear transformation $\mathcal{A}$ on $\mathbb{C}^k$ by

$$\mathcal{A}(e_j) = \sum_{i=1}^{k} a_{ij}e_i, \quad 1 \leq j \leq k.$$

Naturally $\mathcal{A}$ further induces linear transformations on the exterior powers $\wedge^* \mathbb{C}^k$, for which we still use the notation.

Let $S \subset \{1, 2, \cdots, k\}$ be a subset of indices. Denote the number of elements in S by m and write

(3.7) $$S = \{s_1, s_2, \cdots, s_m\} \quad \text{with } s_1 < s_2 < \cdots < s_m.$$

Let T be another subset with the same cardinality as S, and write

(3.8) $$T = \{t_1, t_2, \cdots, t_m\} \quad \text{with } t_1 < t_2 < \cdots < t_m.$$

Let $A_{S,T}$ denote the minor of A with row indices in S and column indices in T, that is,

$$A_{S,T} = \det(a_{s_i t_j})_{i,j=1}^{m}.$$

The efficient way of viewing minors is through exterior products. Insisting on ordering the elements increasingly as in (3.7), we define

$$e_S := e_{s_1} \wedge e_{s_2} \wedge \cdots \wedge e_{s_m} \in \wedge^m \mathbb{C}^k.$$

Then the minor $A_{S,T}$ is nothing but the entry of the matrix for the induced transformation on $\wedge^m \mathbb{C}^n$ in the (e_S, e_T) position, that is,

(3.9) $$\mathcal{A}(e_T) = A_{S,T} e_S + \text{other terms}.$$

Note that for the top exterior form, $A \in SL(k, \mathbb{C})$ implies that

(3.10) $$\mathcal{A}(e_1 \wedge \cdots \wedge e_k) = (\det A)(e_1 \wedge \cdots \wedge e_k) = e_1 \wedge \cdots \wedge e_k.$$

Denote the complement of S in $\{1, 2, \cdots, k\}$ by $\bar{S}$. Let ι denote the inversion of the indices in $\{1, 2, \cdots, k\}$, that is,

(3.11) $$\iota(j) = k + 1 - j, \quad 1 \leq j \leq k.$$

The notation $\iota(S)$ denotes the set $\{\iota(s) \,|\, s \in S\}$, where we need to re-order the elements to make them increasing.

Proposition 3.12. *For $A \in Sp(2n, \mathbb{C})$ or $A \in SO(2n+1, \mathbb{C})$ and any two subsets S and T of $\{1, 2, \cdots, 2n(+1)\}$ of the same cardinality, we have*

$$(3.13) \qquad A_{ST} = A_{\iota(\bar{S})\iota(\bar{T})}.$$

On the other hand, if $A \in SL(k, \mathbb{C})$ and (3.13) is satisfied for all S and T with cardinality 1, that is, for all $1 \le s, t \le k$

$$(3.14) \qquad A_{st} = A_{\iota(\bar{s}), \iota(\bar{t})},$$

where $A_{st} = a_{st}$ is the element of A in the (s, t) position, and $A_{\iota(\bar{s}), \iota(\bar{t})}$ is the minor of A after deleting the $\iota(s)$-th row and $\iota(t)$-th column, then the matrix A is either symplectic or orthogonal depending on the parity of k.

Proof. The two cases are treated in the same way with k denoting either $2n$ in the symplectic case or $2n + 1$ in the orthogonal case. We write J for J_k, and denote its corresponding linear transformation by $\mathcal{J}$. Then from (3.3) and (3.4), we have that A is symplectic or orthogonal if and only if $A^{-1} = J^{-1} A^t J$, or in terms of the linear transformations on $\mathbb{C}^k$,

$$(3.15) \qquad \mathcal{A}^{-1} = \mathcal{J}^{-1} \mathcal{A}^t \mathcal{J},$$

where $\mathcal{A}^t$ denote the linear transformation defined by A^t.

We will prove (3.13) in the following form

$$(3.16) \qquad A_{\bar{S}\bar{T}} = A_{\iota(S)\iota(T)}.$$

We denote the common cardinality of S and T by m and order their elements according to (3.7) and (3.8).

As preparations, we compute the following two signs

$$(3.17) \qquad e_{\bar{S}} \wedge e_S = \varepsilon_1 \, e_1 \wedge \cdots \wedge e_k,$$

$$(3.18) \qquad \mathcal{J} e_S = \varepsilon_2 \, e_{\iota(S)}.$$

Such calculations are elementary. First we have $e_{\bar{S}} \wedge e_S = (-1)^{m(k-m)} e_S \wedge e_{\bar{S}}$. To calculate the sign for $e_S \wedge e_{\bar{S}}$, we need to move the $e_1, \cdots, e_{s_1-1}$ over the m elements of e_{s_i}, and so on. Therefore

$$\varepsilon_1 = (-1)^{m(k-m) + (s_1-1)m + (s_2-1-s_1)(m-1) + \cdots + (s_m - 1 - s_{m-1})}$$

$$= (-1)^{m(k-m) + s_1 + \cdots + s_m - (m + \cdots + 1)} = (-1)^{m(k-m) + \sum_{i=1}^{m} s_i - \frac{m(m+1)}{2}}.$$

Since $\mathcal{J} e_j = (-1)^{k-j} e_{\iota(j)}$, and we need a re-ordering of m elements in $\iota(S)$, we have

$$\varepsilon_2 = (-1)^{k - s_1 + \cdots + k - s_m + \frac{m(m-1)}{2}}$$

$$= (-1)^{mk - \sum_{i=1}^{m} s_i + \frac{m(m-1)}{2}}.$$

Note that

$$(3.19) \qquad \varepsilon_1 \varepsilon_2 = 1.$$

Now we compute as follows.

$$(3.20) \qquad \begin{aligned} A_{\bar{S}\bar{T}} \, e_1 \wedge \cdots \wedge e_k &= \varepsilon_1 A_{\bar{S}\bar{T}} \, e_{\bar{S}} \wedge e_S = \varepsilon_1 (\mathcal{A} e_{\bar{T}}) \wedge e_S \\ &= \varepsilon_1 \mathcal{A}(e_{\bar{T}} \wedge (\mathcal{A}^{-1} e_S)) = \varepsilon_1 e_{\bar{T}} \wedge (\mathcal{A}^{-1} e_S), \end{aligned}$$

where we have used (3.9) and (3.10).

Using (3.15), we see that the above can be continued as

$$\varepsilon_1 e_{\bar{T}} \wedge (\mathcal{J}^{-1}\mathcal{A}^t\mathcal{J}e_S) = \varepsilon_1\varepsilon_2 e_{\bar{T}} \wedge (\mathcal{J}^{-1}\mathcal{A}^t e_{\iota(S)})$$

(3.21)
$$= A_{\iota(S)\iota(T)} e_{\bar{T}} \wedge \mathcal{J}^{-1} e_{\iota(T)}$$
$$= (-1)^{(k-1)(k-m)} A_{\iota(S)\iota(T)} e_{\bar{T}} \wedge \mathcal{J}e_{\iota(T)},$$

by (3.19), (3.2) and $e_{\iota(T)} \in \wedge^{k-m}\mathbb{C}^k$.

Adapting (3.18) to $\iota(T)$ and (3.17) to T, we have

$$e_{\bar{T}} \wedge \mathcal{J}e_{\iota(T)} = (-1)^{mk-\sum_{i=1}^m (k+1-t_i)+\frac{m(m-1)}{2}} e_{\bar{T}} \wedge e_T$$
$$= (-1)^{\sum_{i=1}^m t_i + \frac{m(m+1)}{2}} e_{\bar{T}} \wedge e_T$$
$$= (-1)^{m(k-m)} e_1 \wedge \cdots \wedge e_k.$$

Therefore (3.21) can be continued as

(3.22)
$$(-1)^{(k-1)(k-m)} A_{\iota(S)\iota(T)} e_{\bar{T}} \wedge \mathcal{J}e_{\iota(T)} = (-1)^{(k+m-1)(k-m)} A_{\iota(S)\iota(T)} e_1 \wedge \cdots \wedge e_k$$
$$= A_{\iota(S)\iota(T)} e_1 \wedge \cdots \wedge e_k.$$

The combination of (3.20), (3.21) and (3.22) gives (3.16).

For the converse direction, note that (3.14) is just (3.16) with $S = \{\iota(s)\}$ and $T = \{\iota(t)\}$. In this case, combining Eqs. (3.21), (3.22), (3.14), and (3.20), we get

$$\varepsilon_1 e_{\bar{T}} \wedge (\mathcal{J}^{-1}\mathcal{A}^t\mathcal{J}e_S) = \varepsilon_1 e_{\bar{T}} \wedge (\mathcal{A}^{-1}e_S)$$

for all $S = \{\iota(s)\}$ and $T = \{\iota(t)\}$. Therefore

$$A^{-1} = J^{-1}A^t J.$$

By definitions (3.3) and (3.4), we see that A is symplectic or orthogonal if it satisfies (3.14). $\qquad\square$

4. REDUCTION OF TODA SYSTEMS OF TYPES C_n AND B_n TO A_{k-1}

It is well-known [Lez80, ALW13] that the C_n and B_n Toda systems can be treated as the A_{2n-1} and A_{2n} Toda systems with symmetries. More precisely, we will show in this section that the solutions to Toda systems (1.8) of types C_n and B_n correspond to the solutions of Toda systems (4.3) of types A_{2n-1} and A_{2n} with the symmetry condition (4.4) and with the symmetry requirement (4.5). The symmetry corresponds to the outer automorphism of the Lie algebra A_{k-1}, which graphically is represented by the symmetry of its Dynkin diagram.

In this section, we give no details for the verification of the assertions since they are elementary. The B_n case is slightly more complicated than the C_n case. Note that for the B_n case, $\delta_{i,n}$ denotes the Kronecker delta.

Lemma 4.1 (C_n reduction). *The u_i for $1 \le i \le n$ satisfy (1.1) for the C_n Toda system with parameters γ_i for $1 \le i \le n$ iff the $\tilde{u}_i$ for $1 \le i \le 2n-1$ defined by*

$$\tilde{u}_i = \tilde{u}_{2n-i} = u_i, \quad 1 \le i \le n,$$

satisfy (1.1) for the A_{2n-1} Toda system with parameters $\tilde{\gamma}_i$ for $1 \le i \le 2n-1$ defined by

$$\tilde{\gamma}_i = \tilde{\gamma}_{2n-i} = \gamma_i, \quad 1 \le i \le n.$$

Lemma 4.2 (B_n reduction). *The u_i for $1 \leq i \leq n$ satisfy (1.1) for the B_n Toda system with parameters γ_i for $1 \leq i \leq n$ iff the $\tilde{u}_i$ for $1 \leq i \leq 2n$ defined by*

$$\tilde{u}_i = \tilde{u}_{2n+1-i} = u_i + \delta_{i,n} \ln 2, \quad 1 \leq i \leq n,$$

satisfy (1.1) for the A_{2n} Toda system with parameters $\tilde{\gamma}_i$ for $1 \leq i \leq 2n$ defined by

$$\tilde{\gamma}_i = \tilde{\gamma}_{2n+1-i} = \gamma_i, \quad 1 \leq i \leq n.$$

Similar reductions can be done in the other version of the Toda systems (1.8), and we can treat the two types in the same way as reductions of A_{k-1} Toda systems. We have $k = 2n$ in the C_n case, and $k = 2n + 1$ in the B_n case. Now consider the A_{k-1} Toda system (1.8) as

$$(4.3) \qquad \begin{cases} \Delta \tilde{U}_i + 4 \exp \left(\sum_{j=1}^{k-1} a_{ij} \tilde{U}_j \right) = 4\pi \tilde{\alpha}_i \delta_0, & 1 \leq i \leq k - 1 \\ \displaystyle\int_{\mathbb{R}^2} e^{\sum_j a_{ij} \tilde{U}_j} < \infty, & 1 \leq i \leq k - 1, \end{cases}$$

together with the conditions

$$(4.4) \qquad \tilde{\gamma}_i = \tilde{\gamma}_{k-i} = \gamma_i > -1, \quad 1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor,$$

and the requirement

$$(4.5) \qquad \tilde{U}_i = \tilde{U}_{k-i}, \quad 1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor.$$

In the C_n case, under the condition (4.4), the $\tilde{\alpha}_i$ as in (1.9) defined by the $\tilde{\gamma}_j$ using the inverse Cartan matrix of A_{2n-1} for $1 \leq i, j \leq 2n - 1$ are related to the α_i defined by the γ_j using the inverse Cartan matrix of C_n for $1 \leq i, j \leq n$ by

$$(4.6) \qquad \tilde{\alpha}_i = \tilde{\alpha}_{2n-i} = \alpha_i, \quad 1 \leq i \leq n.$$

Then the

$$U_i := \tilde{U}_i, \quad 1 \leq i \leq n$$

satisfy the C_n Toda system (1.8) with parameters α_i for $1 \leq i \leq n$ from (4.6), and all the solutions to the C_n system are obtained in this way.

Now the B_n case. Under the condition (4.4), the $\tilde{\alpha}_i$ as in (1.9) defined by the $\tilde{\gamma}_j$ using the inverse Cantan matrix of A_{2n} for $1 \leq i, j \leq 2n$ are related to the α_i defined by the γ_j using the inverse Cartan matrix of B_n for $1 \leq i, j \leq n$ by

$$(4.7) \qquad \tilde{\alpha}_i = \tilde{\alpha}_{2n+1-i} = (1 + \delta_{i,n})\alpha_i, \quad 1 \leq i \leq n.$$

Then the

$$(4.8) \qquad U_i := \frac{1}{1 + \delta_{i,n}}(\tilde{U}_i - i \cdot \ln 2), \quad 1 \leq i \leq n$$

satisfy the B_n Toda system (1.8) with parameters α_i for $1 \leq i \leq n$ from (4.7), and all solutions to the B_n system are obtained in this way. Note that the negative coefficient vector of $\ln 2$,

$$\left(1, 2, \cdots, n - 1, \frac{n}{2} \right)^t,$$

is the last column vector of the inverse Cartan matrix of B_n (cf. (1.5)).

5. IMPOSING SYMMETRY FOR SOLUTIONS

The classification result of [LWY12] applies to the system (4.3) with the symmetry condition (4.4) for the $\tilde{\gamma}_i$. As expected, the symmetry requirement (4.5) for the solutions enforces some group structure, and we carry this out in this section.

We follow [LWY12] and Section 2 to present the solutions of (4.3) with (4.4). The integrand functions in (2.21) become

$$(5.1) \qquad \phi_i(z) = \phi_{k-i}(z) = z^{\gamma_i}, \quad \forall 1 \le i \le \left\lfloor \frac{k}{2} \right\rfloor.$$

Then (2.23) becomes $\xi(z) = z^{\tilde{\alpha}_1} = z^{\alpha_1}$ by (4.6) and (4.7). Define $\sigma_i(z)$ by (2.22) and ν_i by (2.24) for $0 \le i \le k - 1$. The vector of functions (2.29) becomes

$$(5.2) \qquad \boldsymbol{\nu} = (\nu_0, \nu_1, \cdots, \nu_{k-1})^t.$$

Remark 5.3. More concretely, in the C_n case the integrand functions and the $\xi(z)$ are

$$(\phi_1, \cdots, \phi_{2n-1}) = (z^{\gamma_1}, \cdots, z^{\gamma_n}, \cdots, z^{\gamma_1}),$$
$$\xi(z) = z^{\gamma_1 + \cdots + \gamma_{n-1} + \frac{1}{2}\gamma_n}.$$

In the B_n case they are

$$(\phi_1, \cdots, \phi_{2n}) = (z^{\gamma_1}, \cdots, z^{\gamma_n}, z^{\gamma_n}, \cdots, z^{\gamma_1}),$$
$$\xi(z) = z^{\gamma_1 + \cdots + \gamma_{n-1} + \gamma_n}.$$

Then the solution to (4.3) is defined by (2.26) which becomes

$$(5.4) \qquad e^{-\tilde{U}_1} = \sum_{i,j=0}^{k-1} h_{ij} \bar{\nu}_i \nu_j,$$

where $H = (h_{ij})_{i,j=0}^{k-1}$ is Hermitian and has determinant 1.

Proposition 5.5. *Formula* (5.4) *defines a solution of* (4.3) *with* (4.4) *that satisfies the requirement* (4.5) *if and only if the Hermitian matrix H is symplectic or orthogonal as in* (3.3) *or* (3.4).

We first do some preparations for the proof. Let $W = W(\boldsymbol{\nu})$ denote the Wronskian matrix of $\boldsymbol{\nu}$ with respect to z, that is,

$$(5.6) \qquad W = \begin{pmatrix} \boldsymbol{\nu} & \boldsymbol{\nu}' & \cdots & \boldsymbol{\nu}^{(k-1)} \end{pmatrix}.$$

We know that $W \in SL(k, \mathbb{C})$ by (2.19). Now consider the big matrix

$$(5.7) \qquad R := W^\dagger H W$$

Then (5.4) says that

$$(5.8) \qquad e^{-\tilde{U}_1} = R_{1,1}.$$

The general theory [Lez80, Nie12, LWY12] of Toda system (4.3) says that in general for $1 \le m \le k - 1$, we have

$$(5.9) \qquad e^{-\tilde{U}_m} = R_{\underline{m},\underline{m}},$$

the first principal minor of R of rank m where $\underline{m}$ denotes the set $\{1, \cdots, m\}$.

Our way to prove Proposition 5.5 is by applying Proposition 3.12 in both directions. First we want to show that the Wronskian matrix in (5.6) has special properties and, loosely speaking, is half symplectic or orthogonal.

Proposition 5.10. [Nie12, Prop. 5.13] *Let $\boldsymbol{\nu}^{(i)}$ denote the ith derivative of $\boldsymbol{\nu}$ (5.2). Then*

$$(5.11) \qquad\qquad (\boldsymbol{\nu}^{(i)})^t J \boldsymbol{\nu}^{(j)} = 0, \qquad \text{if } i + j < k - 1,$$

$$(5.12) \qquad\qquad (\boldsymbol{\nu}^{(i)})^t J \boldsymbol{\nu}^{(j)} = (-1)^i, \qquad \text{if } i + j = k - 1,$$

where $J = J_k$ is as in (3.1).

Remark 5.13. The above proposition is proved in [Nie12] as some differential property of iterated integrals. In [Nie12, Prop. 5.13], the n is our $k - 1$ here, the F^t is our $\boldsymbol{\nu}$, and the swap function s has no effect in our case by our symmetry condition (5.1).

Furthermore, by differentiating (5.12) and by a quick induction, we see that

$$(5.14) \qquad\qquad (\boldsymbol{\nu}^{(i)})^t J \boldsymbol{\nu}^{(j)} = 0, \qquad \text{if } i + j = k.$$

See [Nie12, Eqs. (4.11),(4.14)].

Therefore for the Wronskian matrix W (5.6), we have that

$$(5.15) \qquad P := W^t J W = \begin{pmatrix} & & & & 1 \\ & & & -1 & 0 \\ & & 1 & 0 & p_{2,k-1} \\ & \iddots & \iddots & \iddots & \vdots \\ (-1)^{k-1} & 0 & p_{k-1,2} & \cdots & p_{k-1,k-1} \end{pmatrix},$$

where $p_{i,j} = (\boldsymbol{\nu}^{(i)})^t J \boldsymbol{\nu}^{(j)}$. The P is skew-symmetric or symmetric depending on k.

A Gram-Schmidt procedure can be applied to the column vectors of the Wronskian matrix W to make it symplectic or orthogonal.

Lemma 5.16. *There exists a unipotent upper-triangular matrix U such that*

$$(WU)^t J (WU) = J.$$

Proof. There are several ways of doing the normalization. One particularly nice way is by the following induction scheme.

We first want to get a normalized basis for the plane spanned by the first and the last vectors, i.e. by $\boldsymbol{\nu}$ and $\boldsymbol{\nu}^{(k-1)}$. By (5.11) and (5.12), the only non-normalized quantity is $p_{k-1,k-1} = (\boldsymbol{\nu}^{(k-1)})^t J \boldsymbol{\nu}^{(k-1)}$. This number is nonzero only in the orthogonal case, i.e. when k is odd. Therefore $v_0 = \boldsymbol{\nu}$ and $v_{k-1} = \boldsymbol{\nu}^{(k-1)} - \frac{p_{k-1,k-1}}{2}\boldsymbol{\nu}$ make a normalized basis of this plane.

With respect to this plane, the other vectors may have nonzero pairing

$$(\boldsymbol{\nu}^{(i)})^t J v_{k-1} = (\boldsymbol{\nu}^{(i)})^t J \boldsymbol{\nu}^{(k-1)} = p_{i,k-1}$$

for $1 \le i \le k - 2$ (actually for $2 \le i \le k - 2$ by (5.14)). Then we define $v_i := \boldsymbol{\nu}^{(i)} - p_{i,k-1}\boldsymbol{\nu}$ such that $(v_i)^t J v_{k-1} = 0$ for $1 \le i \le k - 2$.

Then we will go to next plane spanned by v_1 and v_{k-2} and continue this procedure to normalize the vectors v_{k-2} and $v_2, \cdots, v_{k-3}$. Note the sign $(\boldsymbol{\nu}')^t J \boldsymbol{\nu}^{(k-2)} = -1$ from (5.12) will come into the formulas.

Therefore through induction, we have constructed a unipotent upper-triangular matrix U such that WU is symplectic or orthogonal with respect to J. $\qquad\square$

Proposition 5.17. *For any $1 \leq m \leq k$ and any subset S of cardinality m, we have the equality among the minors for the Wronskian matrix* (5.6)

$$W_{S,\underline{m}} = W_{\iota(\bar{S}),\underline{k-m}}.$$

Proof. By Lemma 5.16, the matrix WU is symplectic or orthogonal. It is clear from the definition (3.11) that

$$\iota(\underline{\bar{m}}) = \underline{k-m}.$$

Therefore (3.13) gives that

$$(WU)_{S,\underline{m}} = (WU)_{\iota(\bar{S}),\underline{k-m}}.$$

Since U is a unipotent upper-triangular matrix, WU is obtained from W using column additions by previous columns. Therefore we see that

$$(WU)_{S,\underline{m}} = W_{S,\underline{m}} \quad \text{and} \quad (WU)_{\iota(\bar{S}),\underline{k-m}} = W_{\iota(\bar{S}),\underline{k-m}},$$

since the column indices $\underline{m}$ and $\underline{k-m}$ are strings of numbers starting from 1. $\square$

Now we are ready for the following.

Proof of Proposition 5.5. Actually merely the case of (4.5) when $i = 1$

$$e^{-\tilde{U}_1} = e^{-\tilde{U}_{k-1}}$$

is sufficient for proving that H is either symplectic or orthogonal. By (5.8) and (5.9), we get

(5.18) $$R_{1,1} = R_{\underline{k-1},\underline{k-1}}.$$

By (5.7) and basic properties of determinants, we get

$$R_{1,1} = \sum_{i,j=1}^{k} \overline{W_{i,1}} H_{i,j} W_{j,1},$$

$$R_{\underline{k-1},\underline{k-1}} = \sum_{i,j=1}^{k} \overline{W_{\iota(\bar{i}),\underline{k-1}}} H_{\iota(\bar{i}),\iota(\bar{j})} W_{\iota(\bar{j}),\underline{k-1}},$$

where the second equation uses the fact that any $S \subset \{1, \cdots, k\}$ with $k-1$ elements can be written as the complement of one element $\iota(i)$ for some $1 \leq i \leq k$.

By Proposition 5.17 with $m = 1$ and $S = \{i\}$ or $\{j\}$ for $1 \leq i, j \leq k$, we have

(5.19) $$\overline{W_{i,1}} = \overline{W_{\iota(\bar{i}),\underline{k-1}}}, \quad W_{j,1} = W_{\iota(\bar{j}),\underline{k-1}}.$$

Furthermore, by (5.6), (2.24) and (2.16), we have

(5.20) $$W_{j,1} = \nu_{j-1} = \chi_{j-1}\, z^{\tilde{\mu}_1 + \cdots + \tilde{\mu}_{j-1} - \alpha_1}.$$

where $\tilde{\mu}_i = \tilde{\gamma}_i + 1 > 0$. Therefore the $W_{j,1}$ have strictly increasing exponents. Similarly this holds for the $\overline{W_{i,1}}$. Hence the equalities (5.18) and (5.19) mean that the coefficients are equal, that is,

$$H_{i,j} = H_{\iota(\bar{i}),\iota(\bar{j})}, \quad \forall 1 \leq i, j \leq k.$$

Then (3.14) in Proposition 3.12 implies that H is symplectic when k is even or orthogonal when k is odd.

Furthermore along the same lines, it is easy to see that as long as H is symplectic or orthogonal, the requirement (4.5) is satisfied for all i, that is

$$e^{-\tilde{U}_i} = R_{\underline{i},\underline{i}} = R_{\underline{k-i},\underline{k-i}} = e^{-\tilde{U}_{k-i}},$$

by determinant expansion, Propositions 5.17 and (3.13) of Proposition 3.12. $\square$

As mentioned in Section 2, [LWY12] shows that the Hermitian H is positive definite and hence has the Cholesky decomposition (2.27). Now we show that the corresponding lower-triangular matrix B must also be symplectic or orthogonal according to H.

Proposition 5.21. *For a Hermitian positive-definite symplectic or orthogonal matrix H, consider its Cholesky decomposition $H = B^\dagger B$ where B is lower-triangular. Let $B = \Lambda C$ be the decomposition of B into its diagonal and unipotent parts with $\Lambda = \mathrm{diag}(\lambda_0, \cdots, \lambda_{k-1})$ where $\lambda_i > 0$ and C unipotent. Then C, Λ and hence B are all symplectic or orthogonal.*

Proof. Using $H = B^\dagger B$ in $H^t J H = J$, we get $B^t \bar{B} J \bar{B}^t B = J$, which means

$$(5.22) \qquad\qquad \bar{B} J \bar{B}^t = (B^{-1})^t J B^{-1}.$$

Now B is lower-triangular, and so are $\bar{B}$ and B^{-1}. Also their transposes are upper-triangular. It is easy to see that with respect to the secondary diagonal, the left hand side is lower-triangular while the right is upper-triangular. For the equality to hold, we need that both sides are secondary diagonal.

Plug $B = \Lambda C$ in the right hand side of (5.22) to get $\Lambda^{-1}(C^{-1})^t J C^{-1} \Lambda^{-1}$. Since Λ is diagonal, $(C^{-1})^t J C^{-1}$ is secondary diagonal and is equal to J since C is unipotent. Therefore C^{-1} and hence C are symplectic or orthogonal.

Now (5.22) becomes

$$\Lambda J \Lambda = \Lambda^{-1} J \Lambda^{-1}.$$

This says that Λ^2 is symplectic or orthogonal and hence Λ is. Concretely this means that $\lambda_i \lambda_{k-1-i} = 1$ for $0 \le i \le k - 1$. $\square$

6. The subgroups parametrizing the solutions

The final piece of information is to study the unipotent subgroup N of lower-triangular matrices in $Sp(2n, \mathbb{C})$ or $SO(2n+1, \mathbb{C})$. In this section we will first set up a coordinate system on N. Corresponding to the conditions (2.4) and (2.30) in the A case, there are some further restrictions on N for the solutions to be well-defined on $\mathbb{C}^*$ depending on the γ_i, and we will pin down the restrictions.

The unipotent subgroup N is simply-connected and diffeomorphic to a complex vector space, in particular to its Lie algebra $\mathfrak{n}$. One can use the so-called coordinates of the second type, and we refer the reader to [Kna02, p. 76, Cor. 1.126, Thm. 6.46] for more details. Here we would like to use a slight variation to the coordinates of the second type, which is more convenient for our purposes.

A unipotent lower-triangular matrix, when considered as an element in $SL(k, \mathbb{C})$, can be written as

$$(6.1) \qquad C = \begin{pmatrix} 1 & & & & \\ c_{10} & 1 & & & \\ c_{20} & c_{21} & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ c_{k-1,0} & c_{k-1,1} & \cdots & c_{k-1,k-2} & 1 \end{pmatrix}.$$

For C to be symplectic or orthogonal, it satisfies the condition $C^t J_k C = J_k$.

When $k = 2n$ is even, C is symplectic and we can choose c_{ij} for $0 \leq j < i \leq 2n - 1 - j$, $0 \leq j \leq n - 1$ to be our coordinates on N. The dimension is n^2. The other c's in C can be solved in term of these by the above condition. Similarly when $k = 2n + 1$ is odd, C is orthogonal and we can still choose c_{ij} for $0 \leq j < i \leq 2n - 1 - j$, $0 \leq j \leq n - 1$ to be our coordinates, while the other c's can be solved in term of these.

In this section, we will formulate our results in a Lie-theoretical way. We first introduce a bit more terminology from Lie theory and refer the reader to, for example, [Kna02, FH91]. Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n. Let $\tau_1, \ldots, \tau_n$ be the simple roots, and let Δ^+ denote the set of positive roots of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\tau \in \Delta^+} (\mathfrak{g}_\tau \oplus \mathfrak{g}_{-\tau})$ be the root space decomposition, where $\mathfrak{h}$ is the Cartan subalgebra and $\mathfrak{g}_\tau$ is the root space corresponding to τ generated by a root vector e_τ. Let $\mathfrak{n} = \bigoplus_{\tau \in \Delta^+} \mathfrak{g}_{-\tau}$ be the negative nilpotent Lie algebra. Let G be a Lie group corresponding to $\mathfrak{g}$, and N the subgroup corresponding to $\mathfrak{n}$.

The above coordinates on N in $Sp(2n, \mathbb{C})$ and $SO(2n + 1, \mathbb{C})$ correspond to the roots in the sense that $\frac{\partial}{\partial c_{ij}}\big|_{c=0} C$ in (6.1) is a root vector in the Lie algebra. For completeness, we list the correspondences. The Cartan subalgebra $\mathfrak{h}$ is chosen to consists of diagonal matrices, and L_i denotes the linear function on $\mathfrak{h}$ taking the ith diagonal entry.

The positive roots for C_n are

$$
\begin{cases}
\tau_i + \cdots + \tau_{j-1} = L_i - L_j, & \text{for } 1 \leq i < j \leq n, \\
(\tau_i + \cdots \tau_{n-1}) + (\tau_j + \cdots + \tau_n) = L_i + L_j, & \text{for } 1 \leq i \leq j \leq n.
\end{cases}
$$

There are totally n^2 positive roots. Furthermore, the coordinates c_{ij} in the C_n case correspond to the negative roots as follows. The c_{ij} for $0 \leq j < i \leq n - 1$ corresponds to the negative of $L_{j+1} - L_{i+1}$, and the c_{ij} for $0 \leq j < n \leq i \leq 2n-1-j$ corresponds to the negative of $L_{j+1} + L_{2n-i}$.

The positive roots for B_n are

$$
\begin{cases}
\tau_i + \cdots + \tau_j = L_i - L_{j+1}, & \text{for } 1 \leq i \leq j \leq n, \\
(\tau_i + \cdots \tau_n) + (\tau_j + \cdots + \tau_n) = L_i + L_j, & \text{for } 1 \leq i < j \leq n.
\end{cases}
$$

We note that $L_{n+1} = 0$ for B_n. There are again n^2 positive roots. Furthermore, the coordinates c_{ij} in the B_n case correspond to the negative roots as follows. The c_{ij} for $0 \leq j < i \leq n$ corresponds to the negative of $L_{j+1} - L_{i+1}$, and the c_{ij} for $0 \leq j < n < i \leq 2n - 1 - j$ corresponds to the negative of $L_{j+1} + L_{2n+1-i}$.

Using Proposition 5.21, the formula (5.4) for the solution to the system (4.3) with (4.4) and (4.5) becomes

$$
(6.2) \qquad e^{-\tilde{U}_1} = \boldsymbol{\nu}^\dagger B^\dagger B \boldsymbol{\nu} = |\Lambda C \boldsymbol{\nu}|^2 = \sum_{i=0}^{k-1} \lambda_i^2 \Big| \nu_i + \sum_{j=0}^{i-1} c_{ij} \nu_j \Big|^2,
$$

where the diagonal matrix is $\Lambda = \text{diag}(\lambda_0, \cdots, \lambda_{k-1})$ satisfying $\lambda_i \lambda_{k-1-i} = 1$, and the unipotent matrix C has coordinates as above.

We now study the conditions such that (6.2) is well-defined on $\mathbb{C}^*$. Let $\Gamma = (\gamma_1, \cdots, \gamma_n)$ be the n-tuple with entries $\gamma_i > -1$ from (1.1). We define a subalgebra $\mathfrak{n}_\Gamma$ and a subgroup N_Γ as follows.

Definition 6.3. For $\tau \in \Delta^+$, write $\tau = \sum_{i=1}^n m_i \tau_i$ in terms of the simple roots. Define the number $\tau(\Gamma) = \sum_{i=1}^n m_i \gamma_i$ where we replace τ_i by γ_i.

Define the subset Δ_Γ of Δ^+ as $\Delta_\Gamma = \{\tau \in \Delta^+ \,|\, \tau(\Gamma) \in \mathbb{Z}\}$. Note that Δ_Γ is closed under addition, that is, if $\alpha, \beta \in \Delta_\Gamma$ and $\alpha + \beta \in \Delta^+$, then $\alpha + \beta \in \Delta_\Gamma$.

Define the Lie subalgebra $\mathfrak{n}_\Gamma$ of $\mathfrak{n}$ as $\mathfrak{n}_\Gamma = \bigoplus_{\tau \in \Delta_\Gamma} \mathfrak{g}_{-\tau}$, and the subgroup N_Γ of N as the corresponding Lie subgroup, $N_\Gamma = \exp(\mathfrak{n}_\Gamma)$.

Concretely for the C_n and B_n cases, N_Γ consists of matrices (6.1) where we let a coordinate $c_{ij} = 0$ if the corresponding root does not belong to Δ_Γ.

Inspired by a discussion with Ming Xu on conditions (2.4) and (2.30) in the A case, we give the subgroup N_Γ the following more intrinsic interpretation. There are grading elements $E_j \in \mathfrak{h}$ in the Cartan subalgebra such that $\tau_i(E_j) = \delta_{ij}$, where $\tau_i \in \mathfrak{h}^*$ is regarded as a linear function on $\mathfrak{h}$. Consider the following element in the Cartan subgroup

$$
(6.4) \qquad g_\Gamma := \exp\left(2\pi i \sum_{j=1}^n \gamma_j E_j\right) = \exp\left(2\pi i \sum_{j=1}^n \alpha_j H_j\right),
$$

where $H_j = [e_{\tau_j}, e_{-\tau_j}]$ and $\tau_j(H_j) = 2$. The number $\tau(\Gamma)$ in Definition 6.3 is seen to be $\tau(\Gamma) = \tau(\sum_j \gamma_j E_j)$, and so $\mathrm{Ad}_{g_\Gamma} e_\tau = \exp(2\pi i \tau(\Gamma)) e_\tau$. Hence we see that

$$
(6.5) \qquad \mathfrak{n}_\Gamma = \mathfrak{n}^{\mathrm{Ad}_{g_\Gamma}}, \quad N_\Gamma = N^{\mathrm{Ad}_{g_\Gamma}}
$$

are the fixed point sets of the adjoint actions by g_Γ.

Proposition 6.6. *For the $e^{-\tilde{U}_1}$ in (6.2) to be well-defined on $\mathbb{C}^*$, the matrix C in (6.1) must belong to the subgroup N_Γ.*

Proof. By (2.4) in Theorem 2.1, we see that the coefficient $c_{ij} = 0$ if the $\tilde{\gamma}_{j+1} + \cdots + \tilde{\gamma}_i \notin \mathbb{Z}$. It can be seen that such information is exactly encoded in the root structure of the subgroup N_Γ.

We can also argue using (6.5) as follows. For (6.2) to be well-defined on $\mathbb{C}^*$, we need it to be invariant under the change of $z \mapsto e^{-2\pi i} z$. Using (2.7), (4.6), (4.7), and (6.4), it can be seen that under this change, $\boldsymbol{\nu} \mapsto g_\Gamma \boldsymbol{\nu}$. Therefore the solution (6.2) becomes $\boldsymbol{\nu}^\dagger g_\Gamma^\dagger B^\dagger B g_\Gamma \boldsymbol{\nu}$.

Arguing as in (5.20), we have

$$
B^\dagger B = g_\Gamma^\dagger B^\dagger B g_\Gamma = (g_\Gamma^\dagger B g_\Gamma)^\dagger (g_\Gamma^\dagger B g_\Gamma).
$$

Since $g_\Gamma^\dagger B g_\Gamma$ is still lower-triangular with positive diagonal entries, by the uniqueness of the Cholesky decomposition [GVL96], we see that

$$
g_\Gamma^\dagger B g_\Gamma = B.
$$

Using $B = \Lambda C$, we see that $g_\Gamma^{-1} C g_\Gamma = C$ and $C \in N_\Gamma$ by (6.5). $\qquad \square$

Appendix A. The example of C_3

In the two appendices, we work out the examples of C_3 and B_2 Toda systems to demonstrate our results.

We first consider the C_3 case where we set out to solve the system (1.8) where the Cartan matrix is (1.4) for $n = 3$.

The α in (1.9) are defined in terms of the γ by

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 1 & 2 & 1 \\ 1 & 2 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix},$$

where the matrix is the inverse Cartan matrix of C_3. With $\mu_i = \gamma_i + 1$ for $1 \le i \le 3$ and by Remark 5.3 and (2.17), the $\boldsymbol{\nu}$ has the following shape

$$(A.1) \quad \boldsymbol{\nu} = \frac{1}{z^{\gamma_1 + \gamma_2 + \frac{\gamma_3}{2}}} \Big(1, \frac{z^{\mu_1}}{\mu_1}, \frac{z^{\mu_1 + \mu_2}}{\mu_2(\mu_1 + \mu_2)}, \frac{z^{\mu_1 + \mu_2 + \mu_3}}{\mu_3(\mu_2 + \mu_3)(\mu_1 + \mu_2 + \mu_3)},$$

$$\frac{z^{\mu_1 + 2\mu_2 + \mu_3}}{\mu_2(\mu_2 + \mu_3)(2\mu_2 + \mu_3)(\mu_1 + 2\mu_2 + \mu_3)},$$

$$\frac{z^{2\mu_1 + 2\mu_2 + \mu_3}}{\mu_1(\mu_2 + \mu_1)(\mu_1 + \mu_2 + \mu_3)(\mu_1 + 2\mu_2 + \mu_3)(2\mu_1 + 2\mu_2 + \mu_3)} \Big)^t.$$

By Proposition 5.21, the Λ and C have the following shapes:

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_3^{-1}, \lambda_2^{-1}, \lambda_1^{-1}), \quad \lambda_i > 0,$$

$$C = \begin{pmatrix} 1 & & & & & \\ c_{10} & 1 & & & & \\ c_{20} & c_{21} & 1 & & & \\ c_{30} & c_{31} & c_{32} & 1 & & \\ c_{40} & c_{41} & c_{42} & c_{43} & 1 & \\ c_{50} & c_{51} & c_{52} & c_{53} & c_{54} & 1 \end{pmatrix}$$

where the $c_{51}, c_{52}, c_{42}, c_{53}, c_{43}, c_{54}$ are easily solved in terms of the others by the condition that $C^t J_6 C = J_6$. The results are

$$c_{51} = c_{10}c_{41} - c_{20}c_{31} + c_{30}c_{21} - c_{40},$$

$$c_{42} = c_{21}c_{32} - c_{31}, \quad c_{52} = c_{10}c_{21}c_{32} - c_{10}c_{31} - c_{20}c_{32} + c_{30},$$

$$c_{43} = c_{21}, \quad c_{53} = c_{10}c_{21} - c_{20}, \quad c_{54} = c_{10}.$$

The element in (6.4) is

$$g_\Gamma = \exp\big(2\pi i \, \mathrm{diag}(\alpha_1, \alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \alpha_2 - \alpha_3, \alpha_1 - \alpha_2, -\alpha_1)\big)$$

and the C is required by Proposition 6.6 to satisfy that

$$C g_\Gamma = g_\Gamma C.$$

This restricts some of the free parameters to zero depending on the integrability of the γ. More concretely, we have the following correspondence between the coordinates and the roots

coord's	c_{10}	c_{21}	c_{32}	c_{20}	c_{31}	c_{30}	c_{41}
roots	τ_1	τ_2	τ_3	$\tau_1 + \tau_2$	$\tau_2 + \tau_3$	$\tau_1 + \tau_2 + \tau_3$	$2\tau_2 + \tau_3$
						c_{40}	c_{50}
						$\tau_1 + 2\tau_2 + \tau_3$	$2\tau_1 + 2\tau_2 + \tau_3$

A coordinate must be zero if for the corresponding root $\tau \in \Delta^+$, its value $\tau(\Gamma)$ with the τ_i replaced by γ_i is not an integer.

For example, when $\gamma_1 = -0.5, \gamma_2 = 0.25, \gamma_3 = 1$, the roots with integral values are τ_3 and $\tau_1 + 2\tau_2 + \tau_3$ and the nonzero coordinates are c_{32} and c_{40}.

Appendix B. The example of B_2

For completeness, we also show the solutions to the B_2 Toda system (1.8) as the smallest example of type B. Of course B_2 is isomorphic to C_2, and our result can be compared with those in [ALW13] where it was the C_2 case that was treated.

In this case, the α are defined in terms of the γ by

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix},$$

where the matrix is the inverse Cartan matrix of B_2. With $\mu_i = \gamma_i + 1$ for $1 \le i \le 2$ and by Remark 5.3 and (2.17), the $\boldsymbol{\nu}$ has the following shape

(B.1)
$$\boldsymbol{\nu} = \frac{1}{z^{\gamma_1+\gamma_2}} \left(1, \frac{z^{\mu_1}}{\mu_1}, \frac{z^{\mu_1+\mu_2}}{\mu_2(\mu_1+\mu_2)}, \frac{z^{\mu_1+2\mu_2}}{2\mu_2^2(\mu_1+2\mu_2)}, \frac{z^{2\mu_1+2\mu_2}}{2\mu_1(\mu_1+\mu_2)^2(\mu_1+2\mu_2)} \right)^t.$$

By Proposition 5.21, our Λ and C have the following shapes:

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, 1, \lambda_2^{-1}, \lambda_1^{-1}), \quad \lambda_i > 0,$$

$$C = \begin{pmatrix} 1 & & & & \\ c_{10} & 1 & & & \\ c_{20} & c_{21} & 1 & & \\ c_{30} & c_{31} & c_{32} & 1 & \\ c_{40} & c_{41} & c_{42} & c_{43} & 1 \end{pmatrix}$$

where the $c_{40}, c_{31}, c_{41}, c_{32}, c_{42}, c_{43}$ are easily solved in terms of the others by the condition that $N^t J_5 N = J_5$. The results are

$$c_{40} = c_{10}c_{30} - \frac{1}{2}c_{20}^2,$$

$$c_{31} = \frac{1}{2}c_{21}^2, \quad c_{41} = \frac{1}{2}c_{10}c_{21}^2 - c_{20}c_{21} + c_{30}$$

$$c_{32} = c_{21}, \quad c_{42} = c_{10}c_{21} - c_{20}, \quad c_{43} = c_{10}$$

The element in (6.4) is

$$g_\Gamma = \exp\left(2\pi i \operatorname{diag}(\alpha_1, 2\alpha_2 - \alpha_1, 0, \alpha_1 - 2\alpha_2, -\alpha_1)\right)$$

$$= \exp\left(2\pi i \operatorname{diag}(\gamma_1 + \gamma_2, \gamma_2, 0, -\gamma_2, -\gamma_1 - \gamma_2)\right),$$

and the C is required to satisfy that

$$Cg_\Gamma = g_\Gamma C.$$

This restricts some of the free parameters to zero depending on the if the integrability of the γ. More concretely, we have the following correspondence between the coordinates and the roots

coord's	c_{10}	c_{21}	c_{20}	c_{30}
roots	τ_1	τ_2	$\tau_1 + \tau_2$	$\tau_1 + 2\tau_2$

A coordinate must be zero if for the corresponding roots $\tau \in \Delta^+$, its value $\tau(\Gamma)$ with the τ_i replaced by γ_i is not an integer.

For example, when $\gamma_1 = -0.5$ and $\gamma_2 = 0.25$, the only root with integral value is $\tau_1 + 2\tau_2$ and the nonzero coordinate is c_{30}.

These give the formula (6.2), and the solution U_1 to the B_2 Toda system is $U_1 = \tilde{U}_1 - \ln 2$ by (4.8).

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